Approximating Average Bounded-Angle Minimum Spanning Trees*

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Abstract

Motivated by the problem of orienting directional antennas in wireless communication networks, we study average bounded-angle minimum spanning trees. Let P be a set of points in the plane and let α be an angle. An α-spanning tree (α-ST) of P is a spanning tree of the complete Euclidean graph induced by P such that all edges incident to each point $p \in P$ lie in a fixed wedge of angle α with apex p. An α-minimum spanning tree (α-MST) of P is an α-ST with minimum total edge length.

An average-α-spanning tree (denoted by $\overline{\alpha}$-ST) is a spanning tree with the relaxed condition that incident edges to all points lie in wedges with average angle α. An average-α-minimum spanning tree ($\overline{\alpha}$-MST) is an $\overline{\alpha}$-ST with minimum total edge length.

Let $A(\alpha)$ be the smallest ratio of the length of the $\overline{\alpha}$-MST to the length of the standard MST, over all sets of points in the plane. We investigate bounds for $A(\alpha)$. For $\alpha = \frac{2\pi}{3}$, Biniaz, Bose, Lubiw, and Maheshwari (Algorithmica 2022) showed that $\frac{4}{3} \leq A\left(\frac{2\pi}{3}\right) \leq \frac{3}{2}$. We improve the upper bound and show that $A\left(\frac{2\pi}{3}\right) \leq \frac{13}{9}$. We also study this for $\alpha = \frac{\pi}{2}$ and prove that $\frac{3}{2} \leq A\left(\frac{\pi}{2}\right) \leq 4$.

1 Introduction

A wireless communication network can be represented as a geometric graph in the plane. Each antenna is represented by a point p, its transmission range is represented by a disk with radius r centered at p, and there is an edge between two points if they are within each other's transmission ranges (see Figure 1(a)). The problems of assigning transmission ranges to antennas to achieve networks possessing certain properties has been widely studied [3, 6, 10, 13, 15, 16, 17, 18].

In recent years, there has been considerable research on the problem of replacing omni-directional antennas with directional antennas [1, 2, 5, 7, 9, 11, 12, 14, 15, 19]. Here, the transmission range of each point p is an oriented wedge with apex p and angle α (see Figure 1(b)). Directional antennas provide several advantages over omni-directional antennas, including less potential for interference, lower power consumption, and reduced area where communications could be maliciously intercepted [3, 19].

Motivated by this problem, Aschner and Katz [2] introduced the *α-Spanning Tree* (α-ST): a spanning tree of the complete Euclidean graph in the plane where all incident edges of each point p lie in a fixed wedge of angle α with apex p. With this notation, an α-minimum spanning tree (α-MST) of P is defined to be an α-ST with minimum total edge length. Aschner and Katz [2]

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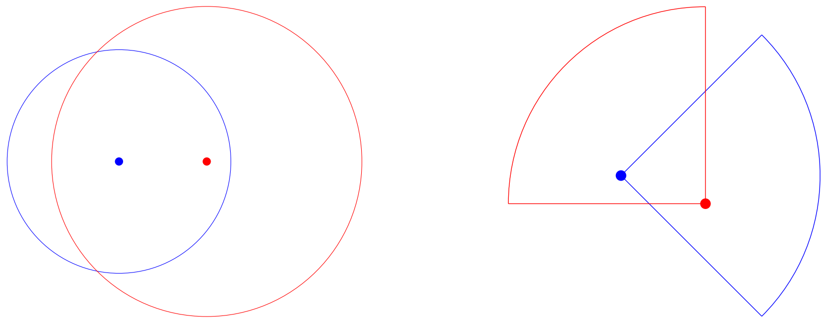

Figure 1: Two points representing antennas, with (a) disks and (b) wedges representing their respective transmission ranges.

showed that the problem of computing an α-MST is NP-hard for the cases where $\alpha = \frac{2\pi}{3}$ and $\alpha = \pi$ by proving reductions to the problem of finding Hamiltonian paths in hexagonal graphs and square grid graphs of degree at most 3, respectively. Biniaz et al. [7] extended this concept to an average-α-minimum spanning tree ($\overline{\alpha}$-MST): an α-MST with the relaxed restriction that the average angle of all the wedges is at most α. More formally, a total angle of αn must be allocated among n points p so that each point has a sufficient allowed angle to cover all incident edges.

The main focus of this paper is computing an $\overline{\alpha}$-ST that approximates an $\overline{\alpha}$-MST for $\alpha = \frac{2\pi}{3}$ and $\alpha = \frac{\pi}{2}$.

1.1 Previous Work

α-MST Problem. Aschner and Katz [2] presented approximation algorithms for the cases where $\alpha = \frac{\pi}{2}, \frac{2\pi}{3}$, and π, with approximation factors of 16, 6, and 2, respectively, with respect to the MST.

The original algorithms of [2] start by computing a MST and then approximating a TSP tour by doubling the MST. For $\alpha = \frac{\pi}{2}$, they arrange the points into consecutive groups of 8. Using a gadget described by Aschner et al. [4], they orient the wedges of the leftmost and rightmost 4 points (of each group) independently such that they are connected, and the combination of the two arrangements covers the entire plane. This also ensures the connectivity of two consecutive groups (because the leftmost 4 points of one group is to the left of the rightmost 4 points of the other group). This gives a 16-approximation of the $\frac{\pi}{2}$-MST. Biniaz et al. [8] improved this to a 10-approximation by showing that an arrangement of groups of 5 points on the TSP tour could be oriented to cover the plane and ensure the connectivity of consecutive groups.

For $\alpha = \frac{2\pi}{3}$, the algorithm of [2] arranges the points of the TSP tour into consecutive groups of 3 points. Then they orient each group independently such that the three points in each group are connected, and their wedges cover the entire plane. They prove that there is a connection between any two independently oriented groups. This ensures the connectivity of the consecutive groups which results in a tree of length at most 6 times the $\frac{2\pi}{3}$-MST. Biniaz et al. [7] improved this approximation ratio to $\frac{16}{3}$ using a similar configuration that takes advantage of the fact that a Hamiltonian path can be made to be non-crossing. Ashur and Katz [5] further improved this to a 4-approximation by showing that any path can be rearranged to a $\frac{2\pi}{3}$-ST with at most twice the weight of the original path. They also showed that 2 is a lower bound for their rearrangement procedure (and thus this approach cannot be improved any further), but note that other approaches might still yield a better approximation factor.

For the $\alpha = \pi$ case, Aschner and Katz [2] showed that the vertices of the TSP tour can be oriented so that any two consecutive points are connected, giving a simple 2-approximation.

For the lower bounds, Aschner and Katz [2] observed that for $\alpha \in \left[\frac{\pi}{3}, \pi\right)$ the α-MST cannot be approximated better than factor 2 with respect to the weight of the MST (e.g. when the points are collinear). For $\alpha \in \left[\pi, \frac{4\pi}{3}\right)$ they observed a lower bound of $\frac{2+\sqrt{3}}{3}$ using four points, with three points forming an equilateral triangle and the fourth point at the center of the triangle.

$\overline{\alpha}$-MST Problem. This problem has previously been studied only for $\overline{\alpha} = \frac{2\pi}{3}$, where Biniaz et al. [7] presented an algorithm that achieves an $\overline{\alpha}$-ST of length at most $\frac{3}{2}$ times the length of the MST. They also proved a lower bound of $\frac{4}{3}$ on the approximation factor with respect to the MST.

1.2 Our Contributions

As the length of the optimal solution is often not known, the length of the underlying MST of the points is usually used as a lower bound. We denote the smallest ratio of the weight of the $\overline{\alpha}$-MST to that of the standard MST over all points in the plane as $A(\alpha)$. Thus the result of [7] implies that $A\left(\frac{2\pi}{3}\right) \leq \frac{3}{2}$. They also showed that $A\left(\frac{2\pi}{3}\right) \geq \frac{4}{3}$.

In this paper, we improve the upper bound on $A\left(\frac{2\pi}{3}\right)$ from $\frac{3}{2}$ to $\frac{13}{9}$ (Section 2). In fact we modify the algorithm of [7] and obtain an $\overline{\alpha}$-ST of weight at most $\frac{13}{9}$ times the weight of the MST. Our algorithm involves a stronger exploitation of the Euclidean metric than the previous work. Our improved upper bound immediately gives an approximation algorithm with ratio $\frac{13}{9}$ (with respect to the MST) for the $\overline{\alpha}$-MST problem for any $\alpha \geq \frac{2\pi}{3}$. For $\alpha = \frac{\pi}{2}$, we present a new approximation algorithm and we show that $\frac{3}{2} \leq A\left(\frac{\pi}{2}\right) \leq 4$ (Section 3). Similar to that of [7], our algorithms run in linear time after computing the MST.

As for the lower bound, Biniaz et al. [7] proved that $A\left(\frac{2\pi}{3}\right) \geq \frac{4}{3}$; this is obtained by points uniformly spaced on a line. We observe that the argument can easily be generalized to obtain a lower bound of $2 - \frac{\alpha}{\pi}$ on $A(\alpha)$ for any $\alpha \leq \pi$, by simply substituting $\frac{2\pi}{3}$ in their argument with α.

1.3 Notation

Throughout this article we use the terms point and vertex interchangeably depending on the context.

To facilitate comparison with previous results, we borrow the following notation from [7]. A *maximal path* in a tree is a path with at least two edges where all internal vertex degrees are 2, and the end vertex degrees are not 2. The *contraction* of a path refers to the removal of all its internal vertices and joining its two endpoints by an edge. The angle that is assigned to a vertex in an $\overline{\alpha}$-ST is called its *charge*—the edges incident to that vertex should lie in a wedge of that angle. We denote the total weight of the edges of a geometric graph G by $w(G)$.

2 A $\frac{13}{9}$-approximation of the $\overline{\frac{2\pi}{3}}$-MST

In this section we focus on the case where $\alpha = \frac{2\pi}{3}$. We first briefly describe the algorithm of Biniaz et al. [7], which operates in two phases by first contracting all maximal paths and then reintroducing them using new "shortcut" edges. By more carefully considering the geometry of the contracted paths, we show how to improve the algorithm by reversing some shortcuts in the second phase to obtain a better approximation ratio.

2.1 The Algorithm of Biniaz et al.

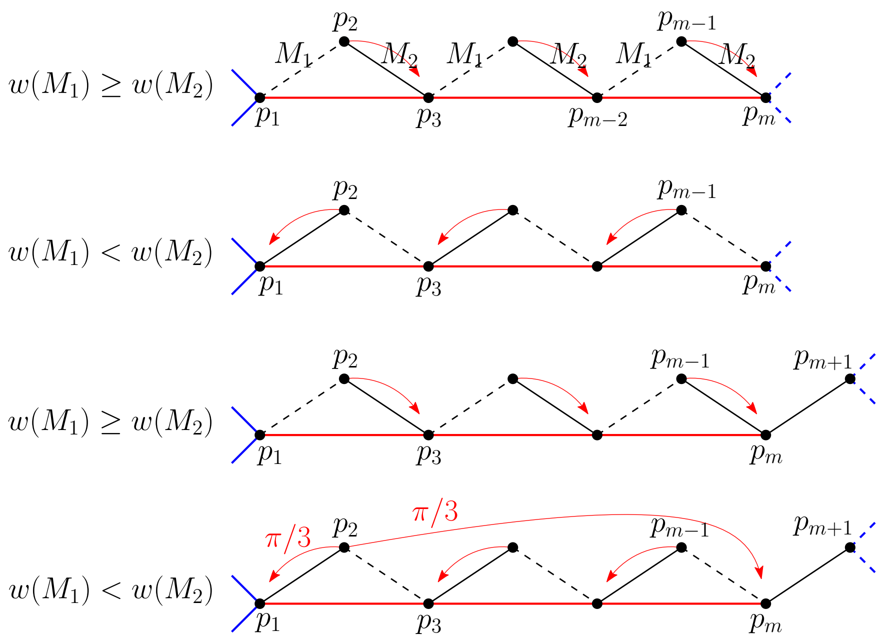

Figure 2: (borrowed from [7]) The contracted path is shown by black segments. The dashed-black edges belong to M_2 and the red edges belong to S.

We begin by briefly describing the algorithm of Biniaz et al. [7], which we refer to as "Algorithm 1". The algorithm starts by computing a minimum spanning tree T of the point set with maximum degree 5 (which is always possible [20]), where each vertex holds a charge of $\frac{2\pi}{3}$. The charges represent the amount of angle allocated to each vertex. The algorithm redistributes the charges among the vertices, but the initial assignment of charges ensures that the sum of the angles assigned to the vertices is $\frac{2\pi n}{3}$ giving an average angle of $\frac{2\pi}{3}$ per vertex. The algorithm goes through two phases that redistribute the charges and also modify the tree. In the first phase, all maximal paths of T are contracted (to edges), resulting in a tree with no vertices of degree 2, and all other vertices having the same degree as in T. We redistribute the charges from degree-1 nodes. Let T be a tree with no nodes of degree 2 and maximum degree 5. Let d_i represent the number of nodes of degree i in T. It is well-known that $d_1 = 2 + d_3 + 2d_4 + 3d_5$. Thus we can redistribute $(i-2)$ charges from degree-1 nodes to nodes of degree $i \in \{3, 4, 5\}$. The charge from the degree-1 nodes (leaves) are redistributed among the internal vertices so that each vertex of degree 3, 4, and 5 has a charge of $\frac{4\pi}{3}$, 2π, and $\frac{8\pi}{3}$, respectively, and still gives us a leftover charge of $\frac{4\pi}{3}$. Since the charge of each internal vertex with degree i is at least $\left(1 - \frac{1}{i}\right) 2\pi$, which covers any set of i edges, all vertices can cover their incident edges. After redistribution, degree-1 vertices have 0 charge and each degree-2 vertex holds its original $\frac{2\pi}{3}$ charge. The leftover $\frac{4\pi}{3}$ charge is split among all leaves in the tree at the end of the algorithm; this is sufficient because the leaves essentially need an angle that is close to zero. In the second phase, the edges of each path $p_1, p_2, \ldots, p_m$ that was contracted in phase 1 are split into two matchings, M_1 and M_2 with equal number of edges. If the path has an odd number of edges then the last edge is not in either matching. The edges of the matching with the larger weight are removed, and a set $S = \{(p_1, p_3), (p_3, p_5), \ldots\}$ of new edges called *shortcuts* is introduced (see Figure 15 of [7], which we include here as Figure 2). By this process, the charge of every new degree-1 vertex is redistributed among other vertices so that each new degree-2 and degree-3 vertex along the path has a charge of π and $\frac{4\pi}{3}$, respectively; this is handled in four cases based on which matching is heavier and whether the path length is even or odd, as shown in Figure 2. Note that the charge given to vertices assigned degree 2 and 3 allows them to cover any set of 2 and 3 edges, respectively.

We also note that both phases can be performed without using the assumption that the underlying MST has maximum degree 5, as each vertex of degree 4 or greater receives at least 2π charge from the leaves it introduces, which covers any number of incident edges.

Let M_1' and M_2' be the union of the edges in the smaller and larger-weight matchings of all contracted paths, respectively. Let T' be the final tree obtained by the above algorithm, and let E be the set of edges of T not in $M_1' \cup M_2'$. Then $w(T) = w(E) + w(M_1') + w(M_2')$. By the triangle equality we have $w(S) \leq w(M_1') + w(M_2')$. Since $w(M_2') \geq w(M_1')$ we get

$$\begin{aligned} w(T') &= w(E) + w(M_1') + w(S) \\ &\leq w(E) + w(M_1') + w(M_1') + w(M_2') \\ &= w(T) + w(M_1') \leq \frac{3}{2} w(T). \end{aligned}$$

2.2 The Improved Algorithm

We begin by modifying the charge-redistribution of phase 2 of Algorithm 1 with a more careful charge redistribution. In particular we show that the 3 edges, that are incident to new degree-3 vertices, can be covered by $\frac{4\pi}{3} - \frac{\pi}{12}$ charge (meaning that we can *save* the $\frac{\pi}{12}$ charge). We then use the saved charge of $\frac{\pi}{12}$ to achieve a better approximation with respect to the original MST. The following lemma, although very simple, plays an important role in the design of the modified algorithm.

Lemma 1. *It is possible to save at least $\frac{\pi}{12}$ charge from every shortcut performed by phase 2 of Algorithm 1.*

Proof. Consider a shortcut ac between two consecutive edges ab and bc of a contracted path as depicted in Figure 3(a). Up to symmetry we may assume that ab is in M_2 and thus it has been removed in phase 2 of Algorithm 1. Denote the angle $\angle bca$ by β. Since the path (a, b, c) is part of the MST, ac is the largest edge of the triangle $\triangle abc$, and thus $\angle abc$ is its largest angle. Therefore $\beta \leq \frac{\pi}{2}$.

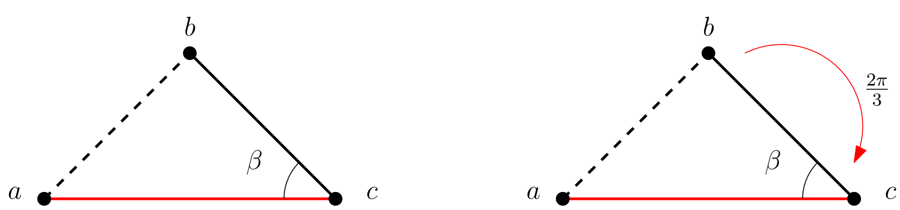

Figure 3: (a) Illustration of a shortcut between the points a and c. (b) Algorithm 1 transfers charge from b to c

The replacement of ab by the shortcut ac has not changed the degree of a, has decreased the degree of b by 1, and has increased the degree of c by 1. Thus the charge assigned to a by Algorithm 1 remains enough to cover its incident edges. Since b has degree 1, its $\frac{2\pi}{3}$ charge is free. Algorithm 1 transfers this free charge to c to cover its new edge (see Figure 3(b)). We show how to cover all edges incident to c while saving $\frac{\pi}{12}$ charge. If c's original degree (i.e. after phase 1 and before phase 2) was 4 or 5 then it carries at least 2π charge which is sufficient to cover its edges. We may assume that the original degree of c is 1, 2, or 3, in which case it holds a charge of 0, $\frac{2\pi}{3}$, or $\frac{4\pi}{3}$, respectively. Thus the new degree of c (after phase 2) is 2, 3, or 4. Based on this we distinguish three cases.

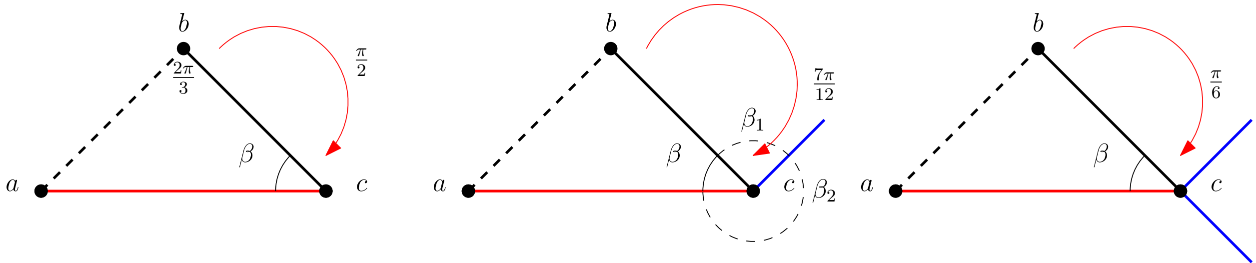

Figure 4: Lemma 1: (a) degree 2 case (b) degree 3 case (c) degree 4 case

- If $\deg(c) = 2$ then the two incident edges of c are ac and bc. We can cover these edges by a charge of β ($\leq \frac{\pi}{2}$). Thus we transfer $\frac{\pi}{2}$ charge from b to c (as shown in Figure 4(a)) and we save $\frac{\pi}{6}$.

- If $\deg(c) = 3$ then we cover β and the smaller of the other two angles at c. Thus the three incident edges to c can be covered by charge of

$$\beta + \left(\frac{2\pi - \beta}{2}\right) = \frac{2\pi + \beta}{2} \leq \frac{2\pi + \frac{\pi}{2}}{2} = \frac{5\pi}{4}.$$

  Thus by transferring $\frac{7\pi}{12}$ from b to c it will have charge of $\frac{5\pi}{4}$ (including its original $\frac{2\pi}{3}$ charge—see Figure 4(b)). Thus we save charge of $\frac{2\pi}{3} - \frac{7\pi}{12} = \frac{\pi}{12}$ from b.

- If $\deg(c) = 4$ then we transfer $\frac{\pi}{6}$ charge from b to c and save the remaining $\frac{\pi}{2}$ charge of b. The vertex c now holds $\frac{3\pi}{2}$ charge (including its charge $\frac{4\pi}{3}$ after phase 1) which covers its four incident edges (see Figure 4(c)).

□

The following is a direct implication of Lemma 1.

Corollary 2. *It is possible to save $\frac{\pi}{3}$ charge from every four shortcuts that are performed by Algorithm 1.*

2.3 Reversing Shortcuts

In this section, we present an approximation algorithm that uses fewer shortcuts than Algorithm 1. In fact the new algorithm reverses a constant fraction of the shortcuts performed by Algorithm 1.

Theorem 3. *Given a set of n points in the plane and an angle $\alpha \geqslant \frac{2\pi}{3}$, there is an $\overline{\alpha}$-spanning tree of length at most $\frac{13}{9}$ times the length of the MST. Furthermore, there is an algorithm to find such an $\overline{\alpha}$-ST that runs in linear time after computing the MST.*

Proof. Let T be a degree-5 minimum spanning tree of the point set, and T' be the $\overline{\frac{2\pi}{3}}$-spanning tree obtained from T by Algorithm 1.

Consider the sequence of shortcuts introduced by Algorithm 1 along each contracted path. Let $s_1, s_2, ..., s_m$ be the concatenation of the sequences for all contracted paths. We split these shortcuts into nine sets $S_0, \dots, S_8$ such that $s_i \in S_{(i \mod 9)}$ for each $i \in \{1, \dots, m\}$. Note that no two adjacent shortcuts in the same contracted path will be in the same set S_i. Moreover the number of shortcuts in any two sets S_i and S_j differs by at most 1. Recall that the edges of each contracted path in Algorithm 1 are split into two matchings M_1 and M_2. Let M_1' be the set of edges that are kept in the tree (i.e. M_1' is the union of the smaller-weight matchings from each contracted path), and let the set of edges in the heavier matchings be M_2'. Among $S_0, \dots, S_8$, let S_8 be the one whose corresponding edges in M_1' (i.e the edges in M_1' adjacent to the shortcut edges) have the largest total weight.

Our plan now is to reverse the shortcuts in S_8, i.e., to replace them by their corresponding edges in M_2'. Let S' be the union of $S_0, \dots, S_7$. Notice that $|S'| \geq 8 \cdot (|S_8| - 1)$ since $|S_i|$ and $|S_j|$ differ by at most 1. Let C denote the pool of charges that is obtained after phase 1 of Algorithm 1, and recall that it contains $\frac{4\pi}{3}$ charge. For each shortcut in S' we reassign the charges between its corresponding points to save at least $\frac{\pi}{12}$ charge (as shown in Lemma 1), and add this charge to C. Thus the total charge of C is at least

$$\frac{4\pi}{3} + 8 \cdot (|S_8| - 1) \cdot \frac{\pi}{12} = (|S_8| + 1) \cdot \frac{2\pi}{3}.$$

We will show that to reverse each shortcut from S_8 it suffices to take $\frac{2\pi}{3}$ charge from C.

Consider any shortcut ac in S_8 between two consecutive edges ab and bc of a contracted path as depicted in Figure 5. We *reverse* this shortcut by replacing ac with the removed edge ab. We also reclaim any portion of b's charge that was transferred to c. Thus the reverse operation brings the charges of b and c back to what it was after phase 1 and before phase 2; in particular it brings the charge of b back to $\frac{2\pi}{3}$. There is one exceptional case where $w(M_1) < w(M_2)$ and the path has an odd number of edges (the last case in Figure 2 where p_3, p_2, p_1 play the roles of a, b, c, respectively). In this case the charge of b (i.e. p_2) would be $\frac{\pi}{3}$ as p_m holds the other $\frac{\pi}{3}$ portion. (Since no two shortcuts in S_8 are adjacent in the same contracted path, we can analyze a reverse operation independently of others. Notice, however, that it is possible that two or more shortcuts of S_8 are adjacent at a vertex that has degree at least 3 after phase 1. In this case, the charge of such a vertex suffices to cover its edges after reversing the shortcuts since it will have at least $\frac{\pi}{6}$ charge added for each new edge introduced by the process described in Lemma 1.) The reverse operation does not change the degree of a and thus its charge remains sufficient to cover its edges. The reverse operation makes b have a degree of 2 and decreases the degree of c by 1.

We take $\frac{\pi}{3}$ charge from C for b to bring it to a charge of π, which covers its two incident edges. If $\deg(c) = 1$ or $\deg(c) \geq 3$, its charge is sufficient to cover its edges. If $\deg(c) = 2$ then we take an additional charge of $\frac{\pi}{3}$ from C for c to cover its two incident edges. In the exceptional case where $w(M_1) < w(M_2)$ and the path has odd number of edges (the last case in Figure 2), $p_2 = b$ holds $\frac{\pi}{3}$ charge, so we take $\frac{2\pi}{3}$ from C for p_2 to cover its two incident edges. Since $p_1 = c$ is of degree 1 or at least 3 (as the contracted path is maximal), its charge (acquired after phase 1) is sufficient to cover its edges. Thus, in the worst case we take $\frac{2\pi}{3}$ from C to reverse every shortcut, which accounts for $|S_8| \cdot \frac{2\pi}{3}$. Thus, $\frac{2\pi}{3}$ charge remains in C.

After reversing all shortcuts in S_8, the pool C is left with at least $\frac{2\pi}{3}$ charge which is distributed among the leaves of the resulting tree.

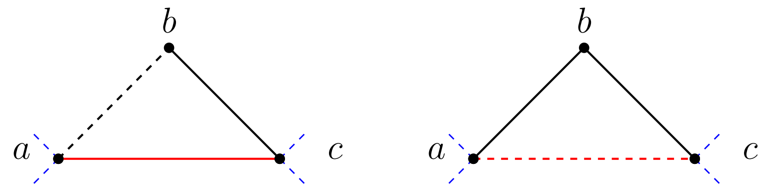

Figure 5: Left: The tree T' before reversing shortcut ac. Right: The tree T'' after reversing ac.

Let T'' be the $\overline{\frac{2\pi}{3}}$-ST tree obtained from T' after reversing all shortcuts in S_8. Let E be the set of edges of T not in $M_1' \cup M_2'$. Let E' be the set of all edges of $M_1' \cup M_2'$ that correspond to the shortcuts in S_8. Let $M_1'' = M_1' \setminus E'$ and $M_2'' = M_2' \setminus E'$ (i.e. all edges in M_1' and M_2', respectively, with a shortcut between their endpoints in T''). Recall that S' is the set of shortcut edges in $S_0, \ldots, S_7$. Then,

$$\begin{aligned}
w(T'') &= w(E) + w(E') + w(S') + w(M_1'') \\
&\leq w(E) + w(E') + w(M_1'') + w(M_2'') + w(M_1'') \\
&= w(T) + w(M_1''),
\end{aligned}$$

where $w(S') \leq w(M_1'') + w(M_2'')$ holds by the triangle inequality. Since S_8 has the largest corresponding M_1' weight, $w(M_1'') \leq \frac{8}{9} w(M_1') \leq \frac{8}{9} \cdot \frac{1}{2} w(T) = \frac{4}{9} w(T)$. Thus,

$$w(T'') \leq w(T) + \frac{4}{9} w(T) = \frac{13}{9} w(T).$$

□

With the lower bound of [7] and the upper bound of Theorem 3 we report the following corollary.

Corollary 4. $\frac{4}{3} \leq A\left(\frac{2\pi}{3}\right) \leq \frac{13}{9}$.

3 A 4-approximation of the $\overline{\frac{\pi}{2}}$-MST

In this section we present an algorithm to compute the $\overline{\frac{\pi}{2}}$-ST. The following theorem summarizes our result.

Theorem 5. *Given a set of n points in the plane and an angle $\alpha \geqslant \frac{\pi}{2}$, there is an $\overline{\alpha}$-spanning tree of length at most 4 times the length of the MST. Furthermore, there is an algorithm to find such an $\overline{\alpha}$-ST that runs in linear time after computing the MST.*

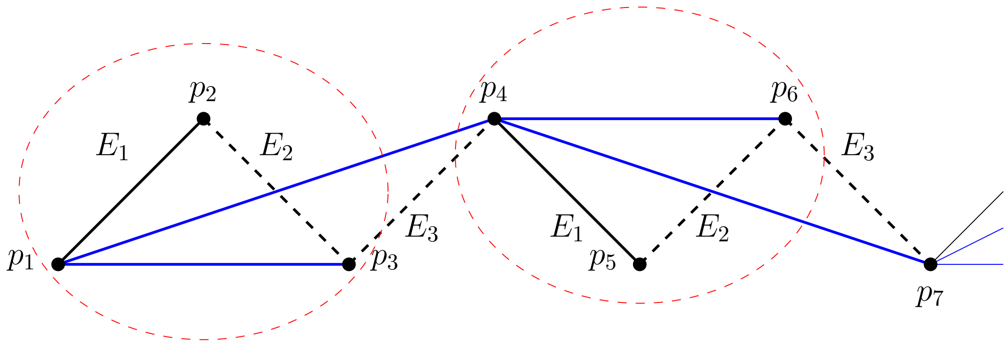

Figure 6: The TSP is in black. Removed edges are dashed, and shortcut edges are in blue.

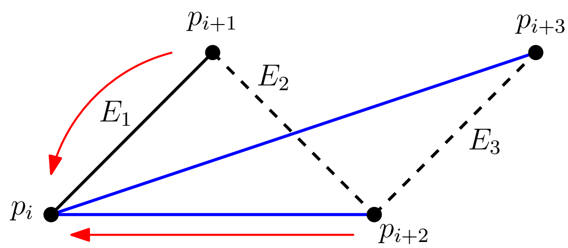

Figure 7: A group of points p_1, p_2, p_3 with shortcuts (blue). Charge is redistributed to p_1.

Proof. Given a set of n points X in the plane, let T be a minimum spanning tree of the complete Euclidean graph induced by X. We begin by computing the metric TSP approximation of T, giving a path $P = p_1, p_2, \ldots, p_n$ with $w(P) \leq 2w(T)$. Let E_1, E_2, and E_3 be the sets obtained by taking every first, second, and third edge of P, respectively (see Figure 6). Without loss of generality assume that the set E_3 has the largest total edge weight. Remove the edges of E_3 from P and obtain a partitioning of the vertices of P into groups of size three (possibly except the first and last groups which may have fewer vertices). Now we consider two cases (i) $w(E_1) \leq w(E_2)$ and (ii) $w(E_1) > w(E_2)$. We describe our proof for case (i); the proof of the other case is analogous.

From now on assume that $w(E_1) \leq w(E_2)$. Remove the edges of E_2 from P. For the three points p_i, p_{i+1}, and p_{i+2} in each group, we transfer the $\pi/2$ charge of each of p_{i+1} and p_{i+2} to p_i as in Figure 7. Then p_i now has total charge $\frac{3\pi}{2}$, which is sufficient to cover any set of four edges incident to p_i. Connect p_{i+2} and p_{i+3} (from the next group) to p_i. This gives a tree T' in which p_i's have degree 4 and other vertices have degree 1 (see Figures 6 and 7). The total length of the edges of type (p_i, p_{i+2}) is at most $w(E_1) + w(E_2)$ by the triangle inequality, and the total length of the edges of type (p_i, p_{i+3}) is at most $w(E_1) + w(E_2) + w(E_3)$. So the total length of these edges of T' is at most $3w(E_1) + 2w(E_2) + w(E_3)$. Thus

$$\begin{aligned}
w(T') &\leq 3w(E_1) + 2w(E_2) + w(E_3) \\
&= (w(E_1) + w(E_2) + w(E_3)) + (w(E_1) + w(E_2)) + w(E_1) \\
&= w(P) + (w(E_1) + w(E_2)) + w(E_1)
\end{aligned}$$

Since E_3 is the set of edges with largest weight, $w(E_3) \geq \frac{1}{3}w(P)$, and thus $w(E_1)+w(E_2) \leq \frac{2}{3}w(P)$.

Similarly, since E_1 has smallest total edge weight, $w(E_1) \leq \frac{1}{3}P$. Thus,

$$\begin{aligned} w(T') &\leq w(P) + \frac{2}{3}w(P) + \frac{1}{3}w(P) \\ &= 2w(P) \\ &\leq 4w(T) \end{aligned}$$

$\square$

Combining the result of Theorem 5 and the lower bound of Section 1.2 for $\alpha = \frac{\pi}{2}$ (that is obtained by a generalization of a result of [7]) we have the following corollary for $A\left(\frac{\pi}{2}\right)$.

Corollary 6. $\frac{3}{2} \leq A\left(\frac{\pi}{2}\right) \leq 4$.

4 Conclusion

In this paper, we have presented approximation algorithms for the $\overline{\alpha}$-MST in the cases where $\alpha = \frac{\pi}{2}$ and $\frac{2\pi}{3}$. Alternatively we obtained the following bounds: $\frac{3}{2} \leq A\left(\frac{\pi}{2}\right) \leq 4$ and $\frac{4}{3} \leq A\left(\frac{2\pi}{3}\right) \leq \frac{13}{9}$.

The obvious open problems are that the bounds on $A\left(\frac{\pi}{2}\right)$ and $A\left(\frac{2\pi}{3}\right)$ are not tight. These could be improved by developing new algorithms with better approximation factors, or finding new sets of points whose $\overline{\alpha}$-MST must have a larger minimum weight with respect to the underlying MST. In particular, we believe that the upper bounds on $A\left(\frac{2\pi}{3}\right)$ and $A\left(\frac{\pi}{2}\right)$ can be further improved.

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